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AdS_3 black holes and a stringy exclusion principle

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Abstract: The duality relating near-horizon microstates of black holes obtained as orbifolds of a subset of AdS_3 to the states of a conformal field theory is analyzed in detail. The $SL(2,R)_L \otimes SL(2,R)_R$ invariant vacuum on AdS_3 corresponds to the NS-NS vacuum of the conformal field theory. The effect of the orbifolding is to produce a density matrix, the temperature and entropy of which coincide with the black hole. For string theory examples the spectrum of chiral primaries agrees with the spectrum of multi-particle BPS states for particle numbers less than of order the central charge. An upper bound on the BPS particle number follows from the upper bound on the $U(1)$ charge of chiral primaries. This is a stringy exclusion principle which cannot be seen in perturbation theory about AdS_3.

Keywords: D-branes, Black Holes in String Theory, Black Holes, Supersymmetry an d Duality.

JHEP12(1998)005

Contents

JHEP12(1998)005

1. Introduction

String theory leads to a surprising and computationally powerful synthesis between gravity in (D+1)-dimensional anti-deSitter space and conformal field theory in D dimensions [1]. The exact duality discovered in [1] relating string theory $AdS_3 \times S^3 \times M^4$, where M^4 is $K3$ or T^4, and two-dimensional conformal field theory on a symmetric product of M^4, provides a rich but highly tractable example (investigated earlier in [2, 3, 4, 5, 6]) which shall be the focus of this paper. The richness follows in part from the fact that black holes can be obtained as orbifolds of a subset of AdS_3 [7]. Both sides of the equation are also relatively tractable theories: the conformal field theory in question is hyperkahler, while on the string theory side one encounters $SL(2,R)$ and $SU(2)$ WZW models. This contrasts with the higher dimensional cases, which require knowledge of higher dimensional conformal field theories and strings propagating in RR backgrounds.

In the three-dimensional case the duality between conformal field theory and quantum gravity on AdS_3, as well as the central charge of the conformal field theory, was derived some time ago in a more general context from the observation that the asymptotic symmetry group of AdS_3 is generated by left and right Virasoro algebras [8]. The Hilbert space must provide a representation of the algebra and so is that of a conformal field theory. This turns out to be enough information to show that the number of states grows at large energy exactly as expected from the Bekenstein-Hawking entropy formula [9]. One would like to go beyond this and explicitly identify the near-horizon microstates responsible for the entropy.

In this paper we begin the process of identifying these microstates in the case of type II string theory on orbifolds of $AdS_3 \times S^3 \times M^4$. Since AdS_3 is $SL(2,R)$ and S^3 is $SU(2)$, the six dimensional part is string theory on a group manifold [10, 11, 12, 13]. This is the near-horizon geometry of the black hole studied in [14] wherein the black hole was shown to be described by conformal field theory whose target space is a deformation of a symmetric product of copies of M^4. We find that the chiral primaries in this conformal field theory correspond to multi-particle BPS states in string theory carrying S^3 angular momentum. The agreement of the degeneracies at every level (up to the exception noted below) involves detailed properties of the dual theories and does not follow from symmetry consideration alone. Non-BPS excitations are $SL(2,R)_L \otimes SL(2,R)_R$ descendants, while Virasoro descendants are more general multi-particle states.

One of the most fascinating results of the analysis is that the upper bound on the $U(1)$ charge encountered in the conformal field theory construction of chiral primaries translates into an exclusion principle limiting the occupation numbers of bosonic BPS particle modes. The maximum allowed occupation number grows in inverse proportion to the coupling constant, and is proportional to the surface area of the region occupied by the particles in Planck units. Hence the bound is nonperturbative in nature, and cannot be seen from perturbative string theory on $AdS_3 \times S^3 \times M^4$. This result is largely algebraic and follows in the spirit of [8, 9] for any sufficiently supersymmetric quantum theory of gravity on AdS_3. Very similar exclusion principles have been encountered in previous investigations of nonperturbative string physics [15, 16, 17] and we expect it has a general significance.

We also find a beautiful relationship (independent of string theory) between the orbifold procedure used to construct the BTZ black hole from AdS_3 and the density matrix of the conformal field theory. The conformal field theory lives on the cylindrical boundary of AdS_3. The boundary inherits from AdS_3 a preferred set of null coordinates on which the $SL(2,R)_L \otimes SL(2,R)_R$ isometries have the canonical action. The conformal field theory is in the vacuum with respect to these coordinates. In general they differ by an exponential transformation from the coordinates used to define energy and momentum, and in which the discrete identifications act simply. This is exactly the transformation from two-dimensional Minkowski to Rindler co-

JHEP12(1998)005

ordinates, and the subset of AdS_3 entering into the construction corresponds to the Rindler wedge. This accounts for the thermal nature of the mixed quantum state of the black hole. Similar relationships are also found between the dual euclidean representations of the finite temperature partition function.

This paper is organized as follows. In section 2 we relate the description of the lorentzian three-dimensional black hole as a quotient of AdS_3 to the conformal field theory description. In section 3 we consider the euclidean black holes as quotients on euclidean AdS_3 and relate them to similar quotients of conformal field theories. In section 3 we consider in detail the map between states in the conformal field theory on the circle with NS boundary conditions and states in supergravity on AdS_3. We find the states corresponding to chiral primary fields in conformal field theory and discuss the bound on the particle number. In section 4 we make some remarks about the case with Ramond boundary conditions.

As this work was in progress we received references [18, 19, 20] which have some overlap with aspects of this paper.

2. Lorentzian black holes

2.1 The classical D1-D5 black hole

In this subsection we describe the near-horizon geometry and fix our notation. The low-energy action for ten-dimensional type IIB string theory contains the terms,

$$\frac{1}{128\pi^7\alpha'^4}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(R+4(\nabla\phi)^2\right)-\frac{1}{12}H^2\right] \tag{2.1}$$

in the ten-dimensional string frame. H denotes the RR three form field strength, and ϕ is the dilaton. The NS three form, self-dual five form, and second scalar are set to zero. We wish to consider a toroidal compactification to five dimensions with an S^1 of length $2\pi R$ and a T^4 of four-volume $(2\pi)^4 v\alpha'^2$.[1] In notation similar to that of [21], the black hole solution associated to Q_1 D1-branes and Q_5 D5-branes is given in terms of the ten-dimensional variables by

$$\begin{aligned}
e^{-2\phi} &= \frac{1}{g^2}\left(1+\frac{r_5^2}{r^2}\right)\left(1+\frac{r_1^2}{r^2}\right)^{-1}, \qquad H=\frac{2r_5^2}{g}\epsilon_3+2r_1^2 g e^{-2\phi} *_6 \epsilon_3,\\
ds^2 &= \left(1+\frac{r_1^2}{r^2}\right)^{-1/2}\left(1+\frac{r_5^2}{r^2}\right)^{-1/2}\Big[-dt^2+(dx^5)^2\\
&\quad+\frac{r_0^2}{r^2}(\cosh\sigma dt+\sinh\sigma dx^5)^2+\left(1+\frac{r_1^2}{r^2}\right)dx_i dx^i\Big]\\
&\quad+\left(1+\frac{r_1^2}{r^2}\right)^{1/2}\left(1+\frac{r_5^2}{r^2}\right)^{1/2}\left[\left(1-\frac{r_0^2}{r^2}\right)^{-1}dr^2+r^2 d\Omega_3^2\right],
\end{aligned} \tag{2.2}$$

[1]With these conventions, T-duality sends R to α'/R or v to $1/v$, and S-duality sends g to $1/g$.

JHEP12(1998)005

where g is the ten-dimensional string coupling, $*_6$ is the Hodge dual in the six dimensions $x^0, \ldots, x^5$, the r_i are functions of the mass and charges, and ϵ_3 is the volume form on the unit three-sphere. x^5 is periodically identified with period $2\pi R$, x^i, $i = 6, \ldots, 9$, are each identified with period $2\pi v^{1/4} \alpha'^{1/2}$. We are interested in the near-horizon scaling limit of [1], defined by taking $\alpha' \to 0$ with

$$U = \frac{r}{\alpha'}, \qquad U_0 = \frac{r_0}{\alpha'}, \tag{2.3}$$

as well as v and R fixed. In this limit 2.2 reduces to

$$\begin{aligned} e^{-2\phi} &= \frac{Q_5}{g_6^2 Q_1}, \\ \frac{ds^2}{\alpha'} &= \frac{U^2}{\ell^2}(-dt^2 + (dx^5)^2) + \frac{U_0^2}{\ell^2}(\cosh\sigma dt + \sinh\sigma dx^5)^2 \\ &\quad + \frac{\ell^2}{U^2 - U_0^2} dU^2 + \sqrt{\frac{Q_1}{vQ_5}} dx_i dx^i + \ell^2 d\Omega_3^2 \,, \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} Q_1 &= \frac{v}{4\pi^2 g^2 \alpha'} \int e^{2\phi} *_6 H \,, \qquad Q_5 = \frac{1}{4\pi^2 \alpha'} \int H \,, \\ g_6^2 &= \frac{g^2}{v}, \qquad k = Q_1 Q_5 \,, \qquad \ell^2 = g_6 \sqrt{k} \,. \end{aligned} \tag{2.5}$$

The shorthand k is adopted because $Q_1 Q_5$ equals the level of the Kac-Moody superconformal algebra of the dual conformal field theory. The total momentum around the S^1 is given by

$$\frac{n}{R} = \frac{R U_0^2 \sinh 2\sigma}{2 g_6^2} \,. \tag{2.6}$$

All charges are normalized to be integers and taken to be positive.

The extremal limit with non-zero momentum charge is $U_0 \to 0$, $\sigma \to \infty$ with $U_0 e^\sigma$ held fixed.

The energy above the $U_0 = n = 0$ black hole ground state is

$$M = \frac{R U_0^2 \cosh 2\sigma}{2 g_6^2} \,. \tag{2.7}$$

We note that this and other relevant quantities are independent of α', and so remain finite in the scaling limit. Note that the volume of the four-torus in 2.4 is $v_f = Q_1/Q_5$. In fact some of the scalar fields have fixed values depending only on the charges (the fixed scalars) [22]. The value of the six dimensional string coupling g_6, on the other hand is not fixed and has the same value that it had asymptotically in 2.2. Similar observations apply to the $K3$ case.

JHEP12(1998)005

2.2 Black holes as quotients

The three-dimensional part of the near-horizon geometry 2.4 is the BTZ black hole [4]. In this section we review its description as a discrete identification of a portion of $SL(2,R) = AdS_3$ [7, 23, 24].

The x^5, r, t part of the metric is locally AdS_3. To make this manifest define new coordinates

$$w^{\pm} = \frac{\sqrt{U^2 - U_0^2}}{U} e^{2\pi T_{\pm}(x^5 \pm t)}, \qquad y = \frac{U_0}{U} e^{\pi T_+(x^5+t) + \pi T_-(x^5-t)}, \tag{2.8}$$

where we have introduced the left and right temperatures

$$T_{\pm} = \frac{1}{2\pi} \frac{U_0 e^{\pm\sigma}}{\ell^2}. \tag{2.9}$$

In terms of these coordinates the three-dimensional metric is locally[2]

$$ds_3^2 = \frac{\ell^2}{y^2}\left(dy^2 + dw^+ dw^-\right). \tag{2.10}$$

This expression is independent of U_0 and σ. Note that the original $x^5 \pm t$ coordinates, prior to the periodic identification, cover only the Rindler wedge of the $w^{\pm}$ Minkowski space. In the asymptotic region, $y \to 0$. The AdS_3 metric 2.10 has an $SL(2,R)_L \otimes SL(2,R)_R$ isometry group. The $SL(2,R)_L$ action is

$$\begin{aligned} w^+ &\to \frac{aw^+ + b}{cw^+ + d}, \\ w^- &\to w^- + \frac{cy^2}{cw^+ + d}, \\ y &\to \frac{y}{cw^+ + d}, \end{aligned} \tag{2.11}$$

where $ad - bc = 1$. The $SL(2,R)_R$ action is similarly

$$\begin{aligned} w^+ &\to w^+ + \frac{cy^2}{cw^- + d}, \\ w^- &\to \frac{aw^- + b}{cw^- + d}, \\ y &\to \frac{y}{cw^- + d}. \end{aligned} \tag{2.12}$$

Note that 2.11, 2.12 map the boundary $(y = 0)$ to itself and act on the boundary as the usual conformal transformations of 1+1 dimensional Minkowski space. The periodic identification of 2.4

$$x^5 \sim x^5 + 2\pi R \tag{2.13}$$

[2]In this and all subsequent line elements we omit for brevity an overall factor of α' which ultimately cancels in the string worldsheet action and drops out.

JHEP12(1998)005

is generated by an element $P = P_L \otimes P_R$ of $SL(2,R)_L \otimes SL(2,R)_R$. The action of P on $y, w^{\pm}$ follows from the coordinate relation 2.8 as

$$w^{\pm} \sim e^{4\pi^2 R T_{\pm}} w^{\pm}, \qquad y \sim e^{2\pi^2 R(T_+ + T_-)} y\,. \tag{2.14}$$

We can then identify P as the $SL(2,R)_L \otimes SL(2,R)_R$ matrix given by

$$P_R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{2\pi^2 R T_+} & 0 \\ 0 & e^{-2\pi^2 R T_+} \end{pmatrix}, \tag{2.15}$$

$$P_L = \begin{pmatrix} e^{2\pi^2 R T_-} & 0 \\ 0 & e^{-2\pi^2 R T_-} \end{pmatrix}. \tag{2.16}$$

P is uniquely defined only up to conjugation.

The extremal limit with nonzero momentum charge is

$$U_0 \to 0\,, \qquad \sigma \to \infty\,, \qquad U_0 e^{\sigma} = \frac{2g_6\sqrt{n}}{R} \qquad \text{fixed}\,. \tag{2.17}$$

In this limit the coordinate transformation 2.8 to AdS_3 coordinates degenerates. Instead the transformations

$$\begin{aligned} w^+ &= \frac{1}{2\pi T_+} e^{2\pi T_+(x^5+t)}\,, \\ w^- &= (x^5 - t) - \frac{\ell^4 \pi T_+}{U^2}\,, \\ y &= \frac{\ell^2}{U} e^{\pi T_+(x^5+t)}\,, \end{aligned} \tag{2.18}$$

take the extremal metric to the AdS_3 form 2.10. At $r = \infty$, we see that w^- is the same as x^-, but w^+ is exponentially related to x^+. Hence the $x^{\pm}$ coordinates cover half of the $w^{\pm}$ Minkowski space, and only the left movers are in a thermal state. The identification $x^5 \sim x^5 + 2\pi R$ corresponds to

$$\begin{aligned} w^+ &\sim e^{4\pi^2 R T_+} w^+\,, \\ w^- &\sim w^- + 2\pi R\,, \\ y &\sim e^{2\pi^2 R T_+} y\,. \end{aligned} \tag{2.19}$$

The corresponding $SL(2,R)_L \otimes SL(2,R)_R$ identifications for the extremal case are given by

$$P_L = \begin{pmatrix} e^{2\pi\sqrt{n/k}} & 0 \\ 0 & e^{-2\pi\sqrt{n/k}} \end{pmatrix}, \tag{2.20}$$

$$P_R = \begin{pmatrix} 1 & 2\pi R \\ 0 & 1 \end{pmatrix}. \tag{2.21}$$

JHEP12(1998)005

We note that for small R - or very low energies - the identification effectively acts only on the left.

The $M=0$ black hole has in addition to 2.17 $n=0$. The three-dimensional part of the $M=0$ metric ($U_0=0$ in 2.4) is transformed to the AdS_3 form 2.10 by the redefinitions

$$\begin{aligned} w^{\pm} &= x^5 \pm t\,, \\ y &= \frac{\ell^2}{U}\,. \end{aligned} \tag{2.22}$$

The identifications are generated by

$$P_L = \begin{pmatrix} 1 & 2\pi R \\ 0 & 1 \end{pmatrix}, \tag{2.23}$$

$$P_R = \begin{pmatrix} 1 & 2\pi R \\ 0 & 1 \end{pmatrix}. \tag{2.24}$$

2.3 The NS1-NS5 black hole

In this subsection we consider the S-dual case of black holes constructed from Q_1 wrapped fundamental strings, Q_5 NS fivebranes and momentum. This case is of special interest because it is described by a known conformal field theory and hence much can be said about the string spectrum and dynamics. The S-dual of the near-horizon metric 2.4, obtained by multiplying by $e^{-\phi}$, is

$$\begin{aligned} d\hat{s}^2 = &\frac{\hat{v}U^2}{Q_1}(-dt^2+(dx^5)^2)+\frac{\hat{v}U_0^2}{Q_1}(\cosh\sigma dt+\sinh\sigma dx^5)^2 \\ &+\frac{Q_5}{U^2-U_0^2}dU^2+d\hat{x}_i d\hat{x}^i+Q_5 d\Omega_3^2\,, \end{aligned} \tag{2.25}$$

where $\hat{v}=1/g_6^2$ is the volume of the four-torus in the dual metric and $\hat{x}_i \sim \hat{x}_i+2\pi\hat{v}^{1/4}$. The six dimensional dilaton is now a fixed scalar

$$\frac{1}{\hat{v}}e^{2\phi}=\frac{Q_5}{Q_1}\,. \tag{2.26}$$

The coordinate transformation 2.8 then reduces the three-metric to

$$ds_3^2=\frac{Q_5}{y^2}\left(dy^2+dw^+dw^-\right), \tag{2.27}$$

which corresponds to a level Q_5 WZW model on the fundamental string worldsheet and does not involve either Q_1 or n. The generic black hole is then constructed with the $SL(2,R)_L\otimes SL(2,R)_R$ identification 2.15, 2.16.

JHEP12(1998)005

2.4 Conformal field theory

In this subsection we show in the conformal field theory picture that the discrete identifications lead precisely to the density matrix expected to correspond to the black hole.

The six-dimensional black string or five-dimensional black hole is described by a

$$c = 6k = 6Q_1Q_5 \tag{2.28}$$

conformal field theory [14]. This conformal field theory can be thought of as living in the asymptotic boundary $r = \infty$ of the solution 2.4, or equivalently $y = 0$ in the coordinates 2.10 [8]. Let us first consider the black string case with no x^5 identifications. The coordinates used to define energy and momentum in the CFT are the flat coordinates in the asymptotic region of the full spacetime solution 2.2, namely x^5, t. In the generic case of nonzero U_0 the conformal field theory is excited and is not in the x^5, t vacuum. Rather it is in the $SL(2,R)_L \otimes SL(2,R)_R$ vacuum with respect to the $w^\pm$ coordinates defined in 2.8. At $r = \infty$ these coordinates are related to x^5, t coordinates via

$$w^\pm = e^{2\pi T_\pm (x^5 \pm t)}\,. \tag{2.29}$$

This is precisely the relation between Rindler and Minkowski coordinates in two dimensions. The quantum state is the $w^\pm$ Minkowski vacuum, but the x^5, t coordinates cover only the Rindler wedge. This is a density matrix rather than a pure state because of correlations with the quantum degrees of freedom behind the Rindler horizon. An inertial observer (moving in a straight line in x^5, t coordinates) will detect a thermal bath of particles, with left and right temperatures given by T_+ and T_-. This bath leads to energy densities $T_{\pm\pm}$, given by the usual schwarzian transformation law

$$T_{++} = \frac{k}{2\pi}\sqrt{\frac{\partial w^+}{\partial x^+}}\frac{\partial^2}{\partial x^{+2}}\sqrt{\frac{\partial x^+}{\partial w^+}} = \frac{\pi}{2}kT_+^2\,. \tag{2.30}$$

with $x^\pm = x^5 \pm t$. A similar expression exists for T_{--} in terms of T_-. This agrees (after integrating around the circle) with the semiclassical expression 2.7, 2.9 for the energy above extremality.

In the extremal case, we see from 2.18 that w^- agrees asymptotically with $x^5 - t$ (up to a constant), but w^+ is exponentially related to $x^5 + t$. Hence only the left movers are in a thermal state. For the $T_+ = T_- = 0$ black string there are no exponential transformations and the conformal field theory is in its ground state.

JHEP12(1998)005

3. Euclidean black holes

In the previous section the precise relation between lorentzian black holes and conformal field theory density matrices, both expressed as $SL(2,R)_L \otimes SL(2,R)_R$ quotients, was presented. In this section a precise relation between euclidean partition functions will be found.

3.1 Euclidean solutions

In this subsection we review the euclidean BTZ black hole [25] [3]. For this purpose it is convenient to write the lorentzian near-horizon three-metric in 2.4 as

$$ds_L^2 = -\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2}dt^2 + \frac{r^2\ell^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2 + r^2(d\phi + \frac{r_+r_-}{r^2}dt)^2\,, \quad (3.1)$$

with

$$r_\pm = \pi\ell\left(\tilde{T}_+ \pm \tilde{T}_-\right), \quad (3.2)$$

where the we have defined the R-independent temperatures $\tilde{T}_\pm = RT_\pm$. In terms of these quantities the momentum n (which is angular momentum from the 2+1 perspective) is

$$n = \frac{kr_-r_+}{\ell^2}\,. \quad (3.3)$$

The euclidean black hole is obtained by analytic continuation of 3.1 to imaginary t and J. In terms of

$$\begin{aligned} t &= i\tau\,, \\ n &= -in_E\,, \end{aligned} \quad (3.4)$$

the euclidean metric is

$$ds_E^2 = \frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2\ell^2}d\tau^2 + \frac{r^2\ell^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2 + r^2(d\phi + \frac{r_+(ir_-)}{r^2}d\tau)^2\,. \quad (3.5)$$

For real τ and n_E, ds_E^2, $r_\pm^2$ and r_+ are all real, T_+ and T_- are complex conjugates of one another and r_- is pure imaginary. The entropy is given by

$$S = \frac{2\pi r_+}{4G_3} = 2\pi^2 k(\tilde{T}_+ + \tilde{T}_-)\,, \quad (3.6)$$

where the three-dimensional Newton's constant is $G_3 = \ell/4k$.

The euclidean quantum theory expanded about 3.5 gives a contribution to the partition function for the theory at inverse temperature

$$\frac{1}{\tilde{T}} = \frac{\partial S}{\partial \tilde{M}} = \frac{1}{2}\left(\frac{1}{\tilde{T}_+} + \frac{1}{\tilde{T}_-}\right). \quad (3.7)$$

[3] We are rescaling t by a factor of ℓ as compared to [25].

JHEP12(1998)005

Since the metric is real for imaginary n there is an imaginary angular potential

$$\frac{\Omega}{\tilde{T}} = \frac{\partial S}{\partial n} = \frac{1}{2}\left(\frac{1}{\tilde{T}_+} - \frac{1}{\tilde{T}_-}\right). \tag{3.8}$$

It is convenient to introduce a complex inverse temperature

$$\beta = \frac{1}{\tilde{T}_+}. \tag{3.9}$$

3.2 $SL(2,C)$

Next we wish to represent 3.5 as a quotient of the three dimensional hyperbolic plane H^3. This was worked out in [25]. Defining new coordinates

$$\begin{aligned} w &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \exp\left\{\frac{r_+ + r_-}{\ell}(\phi + i\tau)\right\}, \\ y &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{1/2} \exp\left\{\frac{r_+}{\ell}\phi + \frac{ir_-}{\ell}\tau\right\}, \end{aligned} \tag{3.10}$$

the metric 3.5 takes the form

$$ds^2 = \frac{\ell^2}{y^2}(dwd\bar{w} + dy^2), \qquad y > 0. \tag{3.11}$$

The identifications implied by the thermal ensemble $\phi + i\tau \sim \phi + i\tau + i\beta$ are automatically taken into account by the exponential map. The periodic identification of ϕ implies identifications of 3.11:

$$\begin{aligned} w &\sim e^{4\pi^2/\beta} w, \\ y &\sim e^{2\pi^2/\beta + 2\pi^2/\beta^*} y. \end{aligned} \tag{3.12}$$

These can be represented as an $SL(2,C)$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.13}$$

for complex a, b, c, d obeying $ad - bc = 1$. The identifications 3.12 of 3.11 are generated by

$$H = \begin{pmatrix} e^{2\pi^2/\beta} & 0 \\ 0 & e^{-2\pi^2/\beta} \end{pmatrix}. \tag{3.14}$$

3.3 The black hole partition function

Euclidean partition functions are defined as functional integrals with fixed boundary conditions at infinity. In the case at hand we wish to fix the inverse temperature and angular potential, which are the real and imaginary parts of β. This is implemented

JHEP12(1998)005

by the boundary condition that the asymptotic geometry is a torus with modular parameter

$$\tau = \frac{i\beta}{2\pi} = \frac{i}{2\pi\tilde{T}_+}\,. \tag{3.15}$$

The semiclassical approximation to minus the logarithm of the partition function is then the action of the least action instanton obeying these boundary conditions.

The euclidean black hole with β given by 3.9 provides such an instanton. The action as a function of β is

$$S_{instanton} = -\frac{\pi^2 k}{\beta} - \frac{\pi^2 k}{\bar{\beta}} = -\frac{i\pi k}{2}\left(\frac{1}{\tau} - \frac{1}{\bar{\tau}}\right). \tag{3.16}$$

A second instanton corresponds to a thermal gas in AdS_3. It is constructed as the periodic identification in euclidean time (denoted τ_E in this subsection to avoid confusion with the modular parameter 3.15)

$$\tau_E + i\phi \sim \tau_E + i\phi + \beta \tag{3.17}$$

of the metric 3.5 with $r_+^2 = -1$ and $r_- = 0$ (namely euclidean AdS_3):

$$ds^2 = (r^2+\ell^2)d\tau_E^2 + \ell^2\frac{dr^2}{r^2+\ell^2} + r^2 d\phi^2\,. \tag{3.18}$$

From the action of the isometries in this coordinate system it can be seen that 3.17 is generated by the $SL(2,C)$ element

$$H = \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}. \tag{3.19}$$

The action of this instanton is the (negative) mass of AdS_3 times the euclidean time:

$$S_{instanton} = -\frac{k(\beta+\bar{\beta})}{4} = \frac{i\pi k}{2}(\tau - \bar{\tau})\,. \tag{3.20}$$

Notice that 3.16 and 3.20 are related by the $SL(2,Z)$ transformation $\tau \to -\frac{1}{\tau}$. One might accordingly anticipate an $SL(2,Z)$ family of instantons with actions

$$S_{instanton} = \frac{i\pi k}{2}\left(\frac{a\tau+b}{c\tau+d} - \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right). \tag{3.21}$$

These can be constructed beginning with the $SL(2,C)$ identification

$$H = \begin{pmatrix} e^{i\pi\frac{a\tau+b}{c\tau+d}} & 0 \\ 0 & e^{-i\pi\frac{a\tau+b}{c\tau+d}} \end{pmatrix}. \tag{3.22}$$

Comparing with 3.14 we see that this yields an instanton whose asymptotic torus has modular parameter $\frac{a\tau+b}{c\tau+d}$ in the coordinates 3.11. However this modular

JHEP12(1998)005

parameter can be transformed to τ by an $SL(2, Z)$ transformation of the coordinate $\tau_E + i\phi$. Hence for every identification of the form 3.22 there is a way of identifying the asymptotic torus so that the geometry satisfies the boundary conditions imposed in the functional representation of the partition function at complex inverse temperature β.

In each modular region of the τ plane there is a unique lowest (negative) action instanton. Defining

$$S_{min}(\tau) \equiv \min \left[\frac{i\pi k}{2} \left(\frac{a\tau + b}{c\tau + d} - \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) \right], \tag{3.23}$$

the leading semiclassical approximation to the partition function is

$$Z(\tau) = e^{-S_{min}(\tau)}. \tag{3.24}$$

At low temperatures (large β) the partition function is dominated by 3.20 corresponding to a thermal gas in AdS_3. At higher temperatures there is a transition to the black hole phase in which 3.16 dominates. This is a sharp first order phase transition in the limit $k \to \infty$ [20].

3.4 Conformal field theory partition function

The euclidean conformal field theory lives on the complex plane with coordinates $(z, \bar{z})$. The action of the global $SL(2, C)$ on $(z, \bar{z})$ is

$$z \to \frac{az + b}{cz + d}, \tag{3.25}$$

for complex a, b, c, d obeying $ad - bc = 1$. H^3 corresponds to the $SL(2, C)$ invariant vacuum on the plane. Hence the black hole partition function should be equivalent to a conformal field theory partition function on the plane identified as

$$z \sim e^{4\pi^2/\beta} z\,. \tag{3.26}$$

This is a toroidal partition function, with modular parameter $\tau = i2\pi/\beta$. It is represented in radial quantization as an annulus with inner and outer edges glued together. Note that the fermions are periodic around the radial cycle and antiperiodic around the angular cycle.

To interpret this as a finite-temperature conformal field theory partition function, consider a modular transformation $\tau \to -1/\tau$. One then has a torus with modular parameter $\tau = i\beta/2\pi$. The antiperiodic direction has length $1/\tilde{T}$, as in the standard representation of a thermal partition function at temperature $\tilde{T}$. The imaginary part of τ leads to the anticipated phase of $-\Omega n/\tilde{T}$ for a state with $n = n_L - n_R$. We see that there is a precise correspondence between the element of the $SL(2, C)$

JHEP12(1998)005

isometry group of H^3 used to construct the euclidean black hole and the element of the $SL(2,C)$ conformal group of the plane used to construct the conformal field theory partition function. Notice that the supergravity result 3.23 implies that for large k only the vacuum contributes once we look at the system in the appropriate channel. For example consider the case with $\Omega = 0$ then we have the CFT on the rectangle. The supergravity result 3.23 means that once we take as euclidean time the longest side of rectangle, only the vacuum propagates.

4. The $AdS_3 \leftrightarrow$ conformal field theory map

In this section we study the correspondence between states in AdS and in the conformal field theory. We first start with some general properties which follow from conformal invariance and then we study properties which are more specific to the (4,4) conformal field theories corresponding to type IIB string theory on $AdS_3 \times S^3 \times M^4$, where M^4 is $K3$ or T^4.

4.1 The NS-NS sector

In this section we consider the NS-NS sector with antiperiodic fermions, which turns out to be the simplest case, i.e. we consider the conformal field theory on a circle of radius one with NS boundary conditions for the fermions. This is the state that we get if we map the plane to cylinder without inserting any operator at the origin.

In the spacetime picture the NS-NS vacuum corresponds simply to the AdS_3 vaccum defined with respect to the coordinates [26, 27]

$$\frac{ds^2}{\ell^2} = -\cosh^2\rho d\tau^2 + \sinh^2\rho d\phi^2 + d\rho^2\,. \tag{4.1}$$

The fermions on the cylinder at infinity are antiperiodic because $\phi \to \phi + 2\pi$ is a 2π rotation of the constant-time disc. In contrast the RR vacuum, where the fermions are periodic under $\phi \to \phi + 2\pi$, corresponds to the $M = 0$ black hole where the surfaces of constant time are not a disc. The three-dimensional mass of AdS_3, defined with respect to the RR vacuum, is $M_3 = -\frac{c}{12}$. This is the value expected from the usual conformal field theoretic mass shift formula for the NS-NS vacuum. In the conformal field theory, NS-NS states are in one-to-one correspondence with local operators on the plane. The coordinate transformation mapping 4.1 to 2.10, given in section 5 below, reduces to the map between the plane and (a diamond in) the cylinder on the boundary. Hence we expect a map between the string Hilbert space on $AdS_3 \times S^3 \times M^4$ and operators of the conformal field theory on the plane. The first step is to identify the $SL(2,R)_L \otimes SL(2,R)_R$ representations of the sta-

JHEP12(1998)005

tes/operators. In the coordinates 4.1, with $u = \tau + \phi$, $v = \tau - \phi$, the $SL(2,R)_L$ generators are described by the vector fields

$$
\begin{aligned}
L_0 &= i\partial_u \,, \\
L_{-1} &= ie^{-iu}\left[\frac{\cosh 2\rho}{\sinh 2\rho}\partial_u - \frac{1}{\sinh 2\rho}\partial_v + \frac{i}{2}\partial_\rho\right], \\
L_1 &= ie^{iu}\left[\frac{\cosh 2\rho}{\sinh 2\rho}\partial_u - \frac{1}{\sinh 2\rho}\partial_v - \frac{i}{2}\partial_\rho\right],
\end{aligned}
\tag{4.2}
$$

normalized so that

$$
[L_0, L_{\pm 1}] = \mp L_{\pm 1}\,, \qquad [L_1, L_{-1}] = 2L_0\,. \tag{4.3}
$$

The $SL(2,R)_R$ generators $\bar{L}_0, \bar{L}_{\pm 1}$ are given by a similar expression with $u \leftrightarrow v$. The quadratic Casimir of $SL(2,R)_L$ is

$$
L^2 = \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) - L_0^2\,. \tag{4.4}
$$

The sum of the two Casimirs is

$$
-2(L^2 + \bar{L}^2) = \partial_\rho^2 + 2\frac{\cosh 2\rho}{\sinh 2\rho}\partial_\rho + \frac{1}{\sinh^2\rho}\partial_\phi^2 - \frac{1}{\cosh^2\rho}\partial_\tau^2\,, \tag{4.5}
$$

which is the laplacian on scalar fields times ℓ^2. Therefore a scalar field of mass m has

$$
L^2 + \bar{L}^2 = -m^2\ell^2/2\,, \tag{4.6}
$$

and the conformal algebra can be used to classify the solutions of the wave equation. In the usual manner consider states with weights $(h, \bar{h})$ under $L_0, \bar{L}_0$ so that

$$
L_0|\psi\rangle = h|\psi\rangle\,, \qquad \bar{L}_0|\psi\rangle = \bar{h}|\psi\rangle\,. \tag{4.7}
$$

It follows that

$$
|\psi\rangle = e^{-ihu - i\bar{h}v}F(\rho)\,. \tag{4.8}
$$

Now suppose that $|\psi\rangle$ is a primary state in the sense that $L_1|\psi\rangle = \bar{L}_1|\psi\rangle = 0$. These conditions imply that that $h = \bar{h}$ and that F satisfies

$$
2h\frac{\sinh\rho}{\cosh\rho}F + \partial_\rho F = 0\,, \tag{4.9}
$$

which is solved by

$$
F = const\frac{1}{(\cosh\rho)^{2h}}\,. \tag{4.10}
$$

Demanding that $|\psi\rangle$ represents a scalar of mass m imposes the additional constraint 4.6.

JHEP12(1998)005

Since $L^2 = L_{-1}L_1 - L_0(L_0 - 1)$ we find that this condition implies that the normalizable solutions have

$$h = \bar{h} = \frac{1}{2}\left(1 + \sqrt{m^2\ell^2 + 1}\right). \tag{4.11}$$

Note that massive string states have m^2 of order one and hence correspond to very large L_0 eigenvalues. For a massless scalar particle in the lth partial wave on S^3 one finds that $h = \bar{h} = 1 + l/2$. The lth partial wave transforms in the $(l/2,l/2)$ representation of $SU(2)_L \times SU(2)_R$. As explained in [28] stability analysis on AdS_3 requires that that $m^2\ell^2 \geq -1$ so that the dimension of the corresponding primary operator is $h \geq 1/2$.

Starting from the primary state 4.10 we can generate all other normalizable solutions with Dirichlet boundary conditions at infinity by acting with L_{-1}, $\bar{L}_{-1}$, since starting from any state we can act with L_1, $\bar{L}_1$ and lower its energy. Since the energy should be positive for the state to have positive norm we conclude that at some point we will get a primary field, and the primary field 4.10 is unique. These $SL(2,R)_L \otimes SL(2,R)_R$ descendants all have the same quadratic Casimirs, but higher integral eigenvalues of $L_0, \bar{L}_0$. These states correspond in an obvious way to $SL(2,R)_L \otimes SL(2,R)_R$ descendents of the primary operators. In particular for a massless scalar $h = \bar{h} = 1$, and the primary field corresponds to a $(1,1)$ operator. Each modulus of the conformal field theory should therefore correspond to a massless scalar in AdS_3. We shall see below that this is indeed the case. Conformal field theory operators may be further classified by their R-charges which identifies the S^3 angular momentum of the corresponding state.

4.2 Chiral primaries

The conformal field theory that we are considering has (4,4) supersymmetry. A particularly interesting set of operators are the chiral primaries, for which the preceding considerations lead to a unique identification with a state in $AdS_3 \times S^3 \times M^4$. The conformal field theory that we are dealing with is some deformation of $Sym_k M^4$. To be specific let us consider in the conformal field theory the (a,c) ring with respect to some particular N=2 subalgebra for the $K3$ case. In order to compute the chiral ring we can take the theory to be simply the orbifold $Sym_k M^4$ since it does not depend on the moduli of the conformal field theory. For $k = 1$, the conformal field theory target space is a single copy of $K3$, and the chiral primary fields are constructed from the 24 harmonic (p,q) forms ω^A, for $A = 1, \dots 24$ as

$$\Phi^A = \omega^A_{ab\ldots\bar{a}\bar{b}\ldots}\psi^a(z)\psi^b(z)\cdots\psi^{\bar{a}}(\bar{z})\psi^{\bar{b}}(\bar{z})\cdots \tag{4.12}$$

This set of fields includes the identity. The R charges are just the rank (p,q) which corresponds to twice the spacetime angular momentum. The cohomology of

JHEP12(1998)005

$Sym_k(K3)$ is the k-fold product of the $K3$ cohomology, plus twisted sector contributions from the fixed points of the permutation group [29]. Let us first consider the subset invariant under the permutation group S_k. This can be described by introducing one species of bosonic creation operator α^A_{-1} for each $K3$ cohomology class[4]. The space of symmetric cohomology elements of $(K3)^k$ are then isomorphic to the k particle Hilbert space. When twisted sectors are included, one has additional creation operators α^A_{-n}, $1 \le n \le k$ and considers the Hilbert space at level k. Hence a general operator involves $M \le k$ oscillators, and can be written

$$\prod_{i=1}^{M} \alpha^{A_i}_{-n_i} |0\rangle \,, \tag{4.13}$$

where $\sum n_i = k$. With each oscillator one can associate the rank (p_i, q_i) of its corresponding cohomology element. The R charge of the operator 4.13 is then

$$(P, Q) = (k - M + \sum p_i, k - M + \sum q_i) \,. \tag{4.14}$$

Since these are chiral primaries one also has $(h, \bar{h}) = (\frac{P}{2}, \frac{Q}{2})$.

We now identify the corresponding states of IIB string theory, beginning with the states with small charges. The vacuum corresponds to the unique state with R-charge $(0, 0)$, constructed from k level one identity oscillators.

At charge $(1, 1)$ there are 20 operators constructed from $k - 1$ level one identity oscillators and one level one rank $(1, 1)$ oscillator. There is an additional operator from $k - 2$ level one and one level two identity oscillators. This gives a total of 21 charge $(1, 1)$ oscillators. Since $h = \bar{h} = \frac{1}{2}$ we see from 4.11 that these operators should correspond to states with negative mass $m^2 = -\frac{1}{\ell^2}$ on AdS_3.

Now we turn to the supergravity description of the corresponding states. Compactifying IIB supergravity on $K3$ gives the unique anomaly-free six-dimensional $(0, 2)$ theory with 21 anti-self-dual and 5 self-dual rank three antisymmetric tensor field strengths. Further compactification on $S^3 \times AdS_3$ requires an expectation value for one of the self-dual tensors. In the six dimensional theory we have 5×21 scalars living in $SO(5, 21)/SO(5) \times SO(21)$ which are in the same supersymmetry multiplet as the anti-self-dual field strengths. When we compactify on $AdS_3 \times S^3$ 21 of the scalars acquire fixed expectation values and their fluctuations around this value get a mass. The other $4\times$ 21 scalars are still massless fields. We define the mass as the number appearing in 4.6. All these are scalars already in six dimensions. In addition, we will have scalar fields on AdS_3 coming from the Kaluza-Klein reduction on the sphere. In particular, the above considerations lead us to expect 21 scalar fields on AdS_3 with $m^2\ell^2 = -1$ corresponding to the minimal weight chiral primaries. We shall see below that these fields arise from the Kaluza-Klein reduction of the anti-self-dual field strengths on S^3. In the process we get, as a by product, the 21 fixed

[4]For the T^4 case we have to include also fermionic oscillators corresponding to odd cohomology classes.

JHEP12(1998)005

scalars. Let us denote by H^1 the self dual field strength which is nonzero. All scalar fields are constant and can be rotated by a global SO(5,21) symmetry so that we now expand around the identity element in the coset $SO(5,21)/SO(5)\times SO(21)$. For small fluctuations we can think of the scalars as a 5×21 matrix w^{im}, $m=1,\ldots,y21$. The 4×21 scalars w^{im}, $i\neq 1$ do not mix with anything and lead to massless fields. We denote the anti-dual-field strengths by K^m. For K^m and w^{1m} we get the following equations [30] (the hatted indices run from 0 to 5)

$$K^m = - * K^m\,, \tag{4.15}$$

$$K^m_{\hat\mu\hat\nu\hat\rho} = w^{1m}H^1_{\hat\mu\hat\nu\hat\rho} + F^m_{\hat\mu\hat\nu\hat\rho}\,, \qquad F^m = dB^m\,, \tag{4.16}$$

where B^m is a two form. We also get the equation

$$(\nabla^2_x + \nabla^2_\theta)w^{1m} - \frac{2}{3}H^{1\hat\mu\hat\nu\hat\rho}K^m_{\hat\mu\hat\nu\hat\rho} = 0\,, \tag{4.17}$$

where ∇^2_x and ∇^2_θ are the laplacian on AdS_3 and S^3 respectively. Since the equations are diagonal in the index m we drop it from now on. We can rewrite 4.15 with 4.16 as

$$F_{\mu\nu\rho} + \tfrac{1}{6}\epsilon_{\mu\nu\rho}\epsilon^{ijk}F_{ijk}\tfrac{2}{\ell}w\epsilon_{\mu\nu\rho} = 0\,, \tag{4.18}$$

$$F_{ij\mu} + \tfrac{1}{2}\epsilon_\mu{}^{\nu\rho}\epsilon_{ij}{}^kF_{k\nu\rho} = 0\,, \tag{4.19}$$

where we separate the indices on AdS_3 (greek) and S^3 (latin). We have also used that $\ell H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}$, $\ell H_{ijk} = \epsilon_{ijk}$ and all other components are zero, as follows from the zeroth order Einstein equations. Now we write the following expansions for B, after imposing the gauge fixing conditions $D^iB_{i\mu} = D^iB_{ij} = 0$,

$$\begin{aligned} B_{\mu\nu} &= b^{I_1}_{\mu\nu}Y^{I_1}\,, \\ B_{\mu i} &= b^{I_3}_{\mu}Y^{I_3}_i\,, \\ B_{ij} &= b^{I_1}\epsilon_{ij}{}^kD_kY^{I_1}\,, \end{aligned} \tag{4.20}$$

where we have used scalar spherical harmonics and vector spherical harmonics. Plugging these equations into the second equation in 4.19 yields an equation for b_μ which decouples from the rest of the equations. From the same equation we further conclude

$$b^{I_1}_{\mu\nu} = \epsilon_{\mu\nu}{}^\rho\partial_\rho b^{I_1}\,, \tag{4.21}$$

if $l>0$. If $l=0$ then $b_{\mu\nu} = b = 0$. Using this equation the first equation in 4.19 becomes

$$\ell^2\nabla^2_x b^{I_1} - C(I_1)b^{I_1} + 2w^{I_1} = 0\,, \tag{4.22}$$

where $C(I_1) = l(l+2)$ is the eigenvalue of the Laplacian on the unit three-sphere for the correponding spherical harmonic (and $l = 0,1,2\cdots$). and we have also expanded w in spherical harmonics. Using 4.22 we see that 4.17 becomes

$$\ell^2\nabla^2_x w^{I_1} - C(I_1)w^{I_1} - 8(-C(I_1)b^{I_1} + w^{I_1}) = 0\,. \tag{4.23}$$

JHEP12(1998)005

The two mass eigenvalues are

$$\ell^2 m_-^2 = l(l-2)\,, \qquad \ell^2 m_+^2 = l(l+6)+8\,. \tag{4.24}$$

Using 4.11 the left and right moving weights are

$$\begin{aligned} \Delta_- &= l/2\,, \qquad && l>0\,, \\ \Delta_+ &= (l+4)/2\,, \qquad && l\geq 0\,. \end{aligned} \tag{4.25}$$

We have included the result for $l=0$ which involves only a mode of w. The first branch in 4.25 describes the scalar corresponding to the chiral primary operator and all its partial waves. The second branch corresponds to the fixed scalar.

Hence we conclude that the 21 charge $(1,1)$ chiral primaries correspond to a single quantum in one of the lowest-lying modes of the 21 $\ell^2 m^2=-1$ scalar fields. Excited single particle states can be constructed as $SL(2,R)_L\otimes SL(2,R)_R$ descendants.

Next let us consider the charge (n,n) chiral primaries. There are 20 such operators from $k-n+1$ level one identity oscillators and one rank $(1,1)$ level n oscillators, as well as one from $k-n$ level one identity oscillators and one level $n+1$ identity oscillator. These can be identified as single particle states arising from higher S^3 angular harmonics of the 21 self-dual tensor multiplets 4.25.

In addition for $n\geq 2$ there is a charge (n,n) operator from $k-n+2$ level one identity oscillators and one rank $(2,2)$ level $n-1$ oscillator. The lowest state corresponds to an operator with angular momentum $(1,1)$ under $SU(2)_L\times SU(2)_R$ and therefore conformal weight (1,1). It could come from deformations of the sphere but we have not checked this explicitly. We also expect chiral primaries coming from the $(0,2)$ and $(2,0)$ forms. These correspond in the conformal field theory to the positively-charged $SU(2)_L$ and $SU(2)_R$ currents $J_L^+(z)$ and $J_R^+(z)$. In supergravity we have an $SU(2)_L\times SU(2)_R$ gauge symmetry coming from isometries of S^3. The positively-charged components of the gauge boson correspond to these chiral primary states.

Finally at charge (n,n) there are additional operators which involve multiple oscillators that are not level one identity operators. These correspond in the obvious manner to multi-particle states, where the total particle number is the number of oscillators that are not level one identity oscillators.

4.3 The stringy exclusion principle

For ordinary particle numbers, we saw in the last subsection that the spectrum of chiral primary operators agrees with the expected BPS supergravity spectrum. However it is abruptly terminated when the particle number reaches $2k$. This follows from fermi statistics of the $2k$ fermionic operators used in the construction of the chiral primaries. In general the maximum R-charge for a chiral primary is bounded

JHEP12(1998)005

by unitarity as $Q \leq c/3 = 2k$. This effect can not be seen in IIB-AdS_3 perturbation theory. From that point of view they are free bosons and there would not seem to be a limit on the occupation number of any particular mode.

It is worth emphasizing that the validity of the bound follows from general symmetry considerations and does not depend on string theory. As discussed in [8, 26, 9] any consistent quantum theory of gravity on AdS_3 is a conformal field theory. If it has in addition sufficiently many spacetime supersymmetries it will be a $(2, 2)$ superconformal theory. In general the lowest-charge (except the identity) chiral primary of the conformal field theory will be a single-particle state on AdS_3. A second chiral primary can in general be constructed by squaring this chiral primary. This operator (when it is non-zero) corresponds to two quanta in the same mode on AdS_3. In general the Nth power of the chiral primary corresponds to N particles in the same mode. However as mentioned above it follows from the $\mathcal{N} = 2$ superconformal algebra that the $U(1)$ charge of a chiral primary is bounded by k. Hence for N of order c the chiral primary will vanish, in accord with the exclusion principle.

Similar phenomena have been encountered in other string theoretic contexts. For example consider the formulation of two-dimensional $SU(N)$ gauge theory on a circle as a string theory [16]. This theory has states in which the string winds m times around the circle. However one must have $m < N$ because N powers of the gauge theory flux tube are trivial. This constraint is invisible in the string perturbation theory. Also in the fermionic formulation of the $c = 1$ matrix model [17], excitations of the upper branch of the Fermi sea are abruptly cut off when the lower Fermi sea is encountered. This again is a bound on the number of quanta in a single mode. In the case of the (0,2) six dimensional CFT it was found in [31] that the single particle spectrum of chiral primaries is bounded by the number of branes, and a similar effects occurs for $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills since the trace of more than N matrices can be decomposed as products of traces. However in those cases there does not seem to be a bound for multiparticle states. Closely related observations of a bound on S^3 angular momentum in the context of AdS_3 scattering were made in [15].

4.4 The Virasoro generators

The primary operators of the conformal field theory have $SL(2, R)_L \otimes SL(2, R)_R$ as well as Virasoro descendants. In general these descendants have differing $SL(2, R)_L \otimes SL(2, R)_R$ Casimirs and correspond to multi-particle states on AdS_3. In this subsection we discuss the Virasoro action on AdS_3.

In quantum gravity the canonical generators of diffeomorphisms contain volume integrals of the constraints. These integrals must be augmented by surface terms in order to have the correct algebra. The latter in turn can be expressed as the volume integral of a total divergence. After fixing the gauge appropriately and imposing

JHEP12(1998)005

the constraints the Virasoro generators L_n on AdS_3 can be written as the volume integral [8]

$$L_n = \int d\Sigma^\mu \zeta_n^\nu \hat{T}_{\mu\nu} \,. \tag{4.26}$$

where $\hat{T}$ is the matter stress tensor plus a gravitational stress tensor given by the Einstein tensor minus the total divergence of the surface term[5]. The vector fields ζ_n

$$\begin{aligned}
\zeta_n^+ &= e^{inu^+} \,, \\
\zeta_n^- &= \frac{n^2}{r^2} e^{inu^+} \,, \\
\zeta_n^r &= -\frac{irn}{2} e^{inu^+} \,.
\end{aligned} \tag{4.27}$$

The choice of subleading in $\frac{1}{r}$ corrections is ambiguous, as such terms manifestly do not contribute in the boundary expression for L_n. We have chosen these terms so that the Lie derivative with respect to ζ_n obeys

$$[\mathcal{L}_{\zeta_m}, \mathcal{L}_{\zeta_n}] = i(n-m)\mathcal{L}_{\zeta_{m+n}} \,, \tag{4.28}$$

although a central charge of course appears in the Poisson brackets of the generators 4.26. $\bar{L}_n$ is given by a similar expression with $+ \leftrightarrow -$. The $SL(2,R)_L \otimes SL(2,R)_R$ subgroup generated by $L_0,\ L_{\pm 1},\ \bar{L}_0,\ \bar{L}_{\pm 1}$ is the isometry group of AdS_3 (although the extension of the vector field into the interior differs from 4.2 at subleading order).

The action of the Virasoro generators on a quantum state will mix up states with different particle numbers, because the vacuum itself cannot be annihilated by all the Virasoro generators. In general the vacuum state depends on the coordinate choice used to distinguish positive and negative frequency excitations. The Virasoro generators transform the vacuum defined with respect to a particular coordinate system to a new state which is the vacuum with respect to the new coordinates.

5. The RR sector

The preceding sections have concentrated on the NS-NS sector of the conformal field theory. In this section we first review the map between the cylinder and the plane and then we make a few remarks about the RR sector.

The map between the lorentzian cylinder 4.1 and the plane 2.10 is given by

$$\begin{aligned}
\frac{1}{y} &= \cosh\rho\cos\tau + \sinh\rho\cos\phi \,, \\
t &= y\cosh\rho\sin\tau \,, \\
x &= y\sinh\rho\sin\phi \,,
\end{aligned} \tag{5.1}$$

[5]Since there are no gravitons in three dimensions this could vanish in an appropriate gauge.

JHEP12(1998)005

where $x \pm t = w^{\pm}$. Remembering that $u = \tau + \phi$ and $v = \tau - \phi$ we see that at the boundary $y = 0^{+}$ this change of coordinates becomes $w^{+} = \tan\frac{u}{2}$, $w^{-} = -\tan\frac{v}{2}$, which is the change of coordinates from the plane to (a diamond in) the cylinder. The $SL(2,R)_L$ generators on the plane are

$$\begin{aligned} H_{-1} &= i\partial_{+}\,, \\ H_0 &= i(w^{+}\partial_{+} + \frac{1}{2}y\partial_y)\,, \\ H_1 &= i((w^{+})^2\partial_{+} + w^{+}y\partial_y - y^2\partial_{-})\,. \end{aligned} \tag{5.2}$$

And similarly for $SL(2,R)_R$ in terms of $\bar{H}$ with $+ \leftrightarrow -$. Notice that the generators in 5.2 are related to those in 4.2 by

$$H_0 = \frac{1}{2i}(L_1 - L_{-1})\,, \qquad H_{\pm 1} = L_0 \mp \frac{1}{2}(L_1 + L_{-1}) \tag{5.3}$$

and the scalar wave operator is

$$-2(L^2 + \bar{L}^2) = y^2(4\partial_{+}\partial_{-} + \partial_y^2 - \frac{1}{y}\partial_y)\,. \tag{5.4}$$

Plane wave solutions for the massless scalar field equation, which diagonalize H_{-1}, $\bar{H}_{-1}$, are Bessel functions of the form

$$\Phi = e^{-i\omega t + ipx} y J_1(\sqrt{\omega^2 - p^2}y)\,, \tag{5.5}$$

where we have imposed Dirichlet boundary conditions at $y = 0$ $(U = \infty)$. The RR sector is obtained by compactifying as indicated in 2.23 2.24. This is a transformation generated by $H_{-1} + \bar{H}_{-1}$ which is the momentum along the plane. Notice that now the fermions are naturally periodic around the circle. As mentioned above this gives the $M = 0$ BTZ black hole which has a singularity at $y = \infty$. This is the location of the horizon in the non-compactified picture (with infinitely long D1-D5 branes). The singularity appears because the circle along which we are identifying becomes null (of zero size) when we approach the horizon $(y = \infty)$. The boundary $(y = 0)$ is a cylinder as in the NS-NS case.

The singularity at $y = \infty$ makes the Hilbert space difficult to analyze. As opposed to the NS-NS case now the quantum gravity vacuum in this geometry is expected to be highly degenerate. In fact from the conformal field theory picture we learn that the asymptotic degeneracy should go as $e^{2\pi\sqrt{c_e k/6}}$ where $c_e = 24$ for $K3$ and $c_e = 12$ for T^4 [32]. (These degeneracies are predicted by U-duality from the degeneracy of perturbative string states in heterotic and type II string theory respectively). Furthermore we expect to have an energy gap of the order $\Delta\omega \sim 1/k$ [33]. Note that if we were to look for states in this theory corresponding to scalar fields, as we did for the NS-NS sector, we would find an infinite number of

JHEP12(1998)005

states corresponding to solutions of 5.5 with $p = 0$ and any energy ω, obtained by continuous rescalings. This is not inconsistent since indeed the gap is zero to leading order in $1/k$. It would be interesting however to understand more precisely in the gravity picture the origin of this gap.

It would be interesting to extend this analysis to other cases involving AdS_3 described in [1, 34].

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JHEP12(1998)005

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JHEP12(1998)005